

FACTORS OF SOME LACUNARY q -BINOMIAL SUMS

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ABSTRACT. In this paper, we prove a divisibility result for the lacunary q -binomial sum

$$\sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} (k-r)/c \\ l \end{bmatrix}_{q^c}.$$

1. INTRODUCTION

Suppose that p is a prime. A classical result of Fleck asserts that

$$\sum_{k \equiv r \pmod{p}} (-1)^k \binom{n}{k} \equiv 0 \pmod{p^{\lfloor \frac{n-1}{p-1} \rfloor}}, \quad (1.1)$$

where $\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$ is the floor function. In 1977, Weisman generalized Fleck's congruence to prime power moduli in the following way:

$$\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} \equiv 0 \pmod{p^{\lfloor \frac{n-p^{\alpha-1}}{p^{\alpha-1}(p-1)} \rfloor}}. \quad (1.2)$$

In 2009, with help of ψ -operator in Fontaine's theory of (ϕ, Γ) -modules, Sun [6] and Wan [9] obtained a polynomial-type extension of (1.1) and (1.2):

$$\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} \binom{(k-r)/p^\alpha}{l} \equiv 0 \pmod{p^{\lfloor \frac{n-p^{\alpha-1}-lp^\alpha}{p^{\alpha-1}(p-1)} \rfloor}}. \quad (1.3)$$

On the other hand, motivated by the homotopy exponents of the special unitary group $SU(n)$, Davis and Sun [3, 8] proved another two congruences with a little different flavor:

$$e \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} ((k-r)/p^\alpha)^l \equiv 0 \pmod{p^{\nu_p(\lfloor n/p^\alpha \rfloor!)}}, \quad (1.4)$$

$$\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} \binom{(k-r)/p^\alpha}{l} \equiv 0 \pmod{p^{\nu_p(\lfloor n/p^{\alpha-1} \rfloor!)-l-\nu_p(l!)}}, \quad (1.5)$$

2010 *Mathematics Subject Classification.* Primary 11B65; Secondary 05A30, 05A10.

Key words and phrases. q -binomial coefficient, cyclotomic polynomial, Lucas congruence.

The author is supported by National Natural Science Foundation of China (Grant No. 10901078).

where $\nu_p(x) = \max\{i \in \mathbb{N} : p^i \mid x\}$ is the p -adic order of x . Notice that neither (1.4) nor (1.5) could be deduced from (1.3), though (1.4) and (1.5) are often weaker than (1.3) provided l is small.

In this paper, we shall consider the q -analogues of (1.4) and (1.5). For an integer n , as usual, define the q -integer

$$[n]_q = \frac{1 - q^n}{1 - q}.$$

And define the q -binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q \cdot [k-1]_q \cdots [1]_q}.$$

In particular, we set $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $k < 0$. It is easy to see $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q since

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

Let $\mathbb{Z}[q]$ denote the polynomial ring in q with integral coefficients. Then we have the following q -analogue of (1.5).

Theorem 1.1. *For $n, c \in \mathbb{Z}^+$ and $r, h \in \mathbb{Z}$, the lacunary q -binomial sum*

$$\sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} (k-r)/c \\ l \end{bmatrix}_{q^c}$$

is divisible by

$$\prod_{c|d} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor lc/d \rfloor} \cdot \prod_{\substack{b|c \\ b < c}} \Phi_b(q)^{\lfloor n/b \rfloor - \lfloor r/b \rfloor - \lfloor (n-r)/b \rfloor}$$

over $\mathbb{Z}[q]$, where Φ_d is the d -th cyclotomic polynomial.

Since $\Phi_{p^\alpha}(q) = [p]_{q^{p^\alpha-1}}$ for prime p , we may get

$$\begin{aligned} & \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} (k-r)/p^\alpha \\ l \end{bmatrix}_{q^{p^\alpha}} \\ & \equiv 0 \pmod{\prod_{j=\alpha}^{\infty} [p]_{q^{p^j-1}}^{\lfloor n/p^j \rfloor - \lfloor l/p^{j-\alpha} \rfloor} \cdot \prod_{j=1}^{\alpha-1} [p]_{q^{p^j-1}}^{\lfloor n/p^j \rfloor - \lfloor r/p^j \rfloor - \lfloor (n-r)/p^j \rfloor}}. \end{aligned} \quad (1.6)$$

Note that

$$\nu_p(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor, \quad (1.7)$$

and for $1 \leq j \leq \alpha - 1$

$$\left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{r}{p^j} \right\rfloor - \left\lfloor \frac{n-r}{p^j} \right\rfloor = \left\lfloor \frac{\{r\}_{p^{\alpha-1}} + \{n-r\}_{p^{\alpha-1}}}{p^j} \right\rfloor - \left\lfloor \frac{\{r\}_{p^{\alpha-1}}}{p^j} \right\rfloor - \left\lfloor \frac{\{n-r\}_{p^{\alpha-1}}}{p^j} \right\rfloor,$$

where $\{r\}_{p^{\alpha-1}}$ denotes the least non-negative residue of r modulo $p^{\alpha-1}$. Substituting $q = 1$ in (1.6), we can get the following stronger version of (1.5) [8, (1.1)]:

$$\begin{aligned} & \nu_p \left(\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} \binom{(k-r)/p^\alpha}{l} \right) \\ & \geq \nu_p([n/p^{\alpha-1}]!) - l - \nu_p(l!) + \tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}), \end{aligned} \quad (1.8)$$

where

$$\tau_p(a, b) = \text{ord}_p \left(\binom{a+b}{a} \right).$$

We shall prove Theorem 1.1 in the next section. For the advantage of q -congruences, our proof of Theorem 1.1 is even simpler than the original one of (1.5).

Remark. Quite recently, some Fleck type q -congruences also have been established by Schultz and Walker [5].

2. PROOFS OF THEOREM 1.1

Let $\mathbb{Q}[q]$ denote the polynomial ring in q with rational coefficients. Note that the greatest common divisor of all coefficients of $\Phi_d(q)$ is 1. By a well-known result of Gauss, if $\Phi_d(q)$ divides $F(q) \in \mathbb{Z}[q]$ over $\mathbb{Q}[q]$, then $\Phi_d(q)$ also divides $F(q)$ over $\mathbb{Z}[q]$. So below we don't distinguish the q -congruences over $\mathbb{Z}[q]$ and $\mathbb{Q}[q]$.

Lemma 2.1. $(\zeta^r q^h; q)_n$ is divisible by $\Phi_d(q)^{\lfloor n/d \rfloor}$ for any $r, s \in \mathbb{Z}$, where $\zeta = e^{2\pi\sqrt{-1}/d}$.

Proof. We know that

$$\Phi_d(q) = \prod_{\substack{k=1 \\ (k,d)=1}}^d (1 - \zeta^k q).$$

For any k with $(k, d) = 1$, let $0 \leq e_k < d$ be the integer such that $e_k k \equiv r \pmod{d}$. Then we have $1 - \zeta^r \zeta^{-e_k k} = 0$, i.e., $1 - \zeta^k q$ divides $1 - \zeta^r q^j$ if $j \equiv e_k \pmod{d}$. Thus

$$(\zeta^r q^h; q)_n = \prod_{j=h}^{n+h-1} (1 - \zeta^r q^j)$$

is divisible by $(1 - \zeta^k q)^{\lfloor n/d \rfloor}$. □

Lemma 2.2.

$$\sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n \\ k \end{bmatrix}_q \equiv 0 \pmod{\prod_{c|d} \Phi_d(q)^{\lfloor n/d \rfloor}}. \quad (2.1)$$

Proof. In view of the q -binomial theorem (cf. [2, Corollary 10.2.2(c)]),

$$\sum_{k=0}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x; q)_n.$$

So letting $\zeta = e^{2\pi\sqrt{-1}/c}$,

$$\begin{aligned} \sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{1}{c} \sum_{k=0}^n (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{t=0}^{c-1} \zeta^{(k-r)t} \\ &= \frac{1}{c} \sum_{t=0}^{c-1} \zeta^{-rt} (\zeta^t q^h; q)_n. \end{aligned}$$

Thus (2.1) immediately follows from Lemma 2.1, since ζ is also a d -th root of unity if $c \mid d$. \square

Lemma 2.3.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{1 < d \leq n} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor}.$$

Proof. Clearly

$$[n]_q! = \prod_{j=1}^n [j]_q = \prod_{j=1}^n \prod_{\substack{d>1 \\ d|j}} \Phi_d(q) = \prod_{1 < d \leq n} \Phi_d(q)^{\lfloor n/d \rfloor}. \quad (2.2)$$

Hence

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]! [n-k]_q!} = \prod_{1 < d \leq n} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor}.$$

\square

Proof of Theorem 1.1. We shall prove

$$\sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} (k-r)/c \\ l \end{bmatrix}_{q^c} \equiv 0 \pmod{\prod_{c|d} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor lc/d \rfloor}} \quad (2.3)$$

by using an induction on l . The case $l = 0$ follows from Lemma 2.2. Assume that $l \geq 1$ and (2.3) holds for the smaller values of l . Compute

$$\begin{aligned}
& \sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} (k-r)/c \\ l \end{bmatrix}_{q^c} \\
&= \sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n \\ k \end{bmatrix}_q \cdot \frac{q^{-r}([k]_q - [r]_q)}{[c]_q [l]_{q^c}} \cdot \begin{bmatrix} (k-r)/c - 1 \\ l - 1 \end{bmatrix}_{q^c} \\
&= \frac{q^{-r}[n]_q}{[lc]_q} \sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} (k-r-c)/c \\ l-1 \end{bmatrix}_{q^c} \\
&\quad - \frac{q^{-r}[r]_q}{[lc]_q} \sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} (k-r-c)/c \\ l-1 \end{bmatrix}_{q^c}.
\end{aligned}$$

Note that $[lc]_q$ is divisible by or prime to $\Phi_d(q)$ according to whether $d \mid lc$ or not. By the induction hypothesis, we obtain that

$$\begin{aligned}
& \frac{q^{-r}[n]_q}{[lc]_q} \sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} (k-r-c)/c \\ l-1 \end{bmatrix}_{q^c} \\
&= \frac{q^{s-r}[n]_q}{[lc]_q} \sum_{k \equiv r-1 \pmod{c}} (-1)^k q^{\binom{k}{2} + (h+1)k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} (k-(r-1+c))/c \\ l-1 \end{bmatrix}_{q^c} \\
&\equiv 0 \pmod{\prod_{c \mid d} \Phi_d(q)^{\lfloor (n-1)/d \rfloor + \mathbf{1}_{d \mid n} - \lfloor (l-1)c/d \rfloor - \mathbf{1}_{d \mid lc}}},
\end{aligned}$$

and

$$\begin{aligned}
& \frac{q^{-r}[r]_q}{[lc]_q} \sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} (k-(r+c))/c \\ l-1 \end{bmatrix}_{q^c} \\
&\equiv 0 \pmod{\prod_{c \mid d} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor (l-1)c/d \rfloor - \mathbf{1}_{d \mid lc}}},
\end{aligned}$$

where for an assertion A we adopt the notation $\mathbf{1}_A = 1$ or 0 according to whether A holds or not. Thus by noting that for arbitrary positive integers s and t

$$\mathbf{1}_{t \mid s} = \left\lfloor \frac{s}{t} \right\rfloor - \left\lfloor \frac{s-1}{t} \right\rfloor,$$

(2.3) is concluded.

On the other hand, with help of Lemma 2.3,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \equiv 0 \pmod{\prod_{b \mid c} \Phi_b(q)^{\lfloor n/b \rfloor - \lfloor r/b \rfloor - \lfloor (n-r)/b \rfloor}}$$

whenever $k \equiv r \pmod{c}$, since for any $b \mid c$

$$\left\lfloor \frac{n}{b} \right\rfloor - \left\lfloor \frac{k}{b} \right\rfloor - \left\lfloor \frac{n-k}{b} \right\rfloor = \left\lfloor \frac{n}{b} \right\rfloor - \left\lfloor \frac{r}{b} \right\rfloor - \left\lfloor \frac{n-r}{b} \right\rfloor.$$

All are done. \square

It is easy to check that

$$[l]_{q^c} [l-1]_{q^c} \cdots [2]_{q^c} [1]_{q^c} = \prod_{1 < j \leq l} \Phi_j(q^c)^{\lfloor l/j \rfloor} \equiv 0 \pmod{\prod_{\substack{c \mid d \\ d > c}} \Phi_d(q)^{\lfloor lc/d \rfloor}}$$

and

$$[k]_q [k-1]_q \cdots [k-l+1]_q = q^{-\binom{l}{2}} [k]_q ([k]_q - [1]_q) \cdots ([k]_q - [l-1]_q).$$

So applying a simple induction on l , we can deduce the q -analogue of (1.4):

Corollary 2.1.

$$\sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n \\ k \end{bmatrix}_q [(k-r)/c]_{q^c}^l.$$

is divisible by

$$\Phi_c(q)^{\lfloor n/c \rfloor - l} \prod_{\substack{c \mid d \\ d > c}} \Phi_d(q)^{\lfloor n/d \rfloor} \cdot \prod_{\substack{b \mid c \\ b < c}} \Phi_b(q)^{\lfloor n/b \rfloor - \lfloor r/b \rfloor - \lfloor (n-r)/b \rfloor}.$$

In particular, for prime p ,

$$\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k q^{\binom{k}{2} + hk} \begin{bmatrix} n \\ k \end{bmatrix}_q [(k-r)/p^\alpha]_q^l$$

is divisible by

$$[p]_{q^{p^{\alpha-1}}}^{\lfloor n/p^\alpha \rfloor - l} \prod_{j=\alpha+1}^{\infty} [p]_{q^{p^{j-1}}}^{\lfloor n/p^j \rfloor} \cdot \prod_{j=1}^{\alpha-1} [p]_{q^{p^{j-1}}}^{\lfloor n/p^j \rfloor - \lfloor r/p^j \rfloor - \lfloor (n-r)/p^j \rfloor}.$$

Substituting $q = 1$, we obtain the improvement of (1.4) [3, Theorem 5.1]:

$$\begin{aligned} & \nu_p \left(\sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} ((k-r)/p^\alpha)^l \right) \\ & \geq \max\{\nu_p(\lfloor n/p^{\alpha-1} \rfloor!) - l, \nu_p(\lfloor n/p^\alpha \rfloor!)\} + \tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}). \end{aligned} \quad (2.4)$$

3. LUCAS TYPE AND WOLSTENHOLME-LJUNGGREN TYPE q -CONGRUENCES

Let

$$T_{p^\alpha, l}(n, r) = \frac{l!p^l}{[n/p^{\alpha-1}]!} \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k \binom{n}{k} \binom{(k-r)/p^\alpha}{l}.$$

In [8], Sun and Davis established the following Lucas type congruence:

$$T_{p^{\alpha+1}, l}(pn + s, pr + t) \equiv (-1)^t \binom{s}{t} T_{p^\alpha, l}(n, r) \pmod{p}, \quad (3.1)$$

where p is a prime, $\alpha \geq 1$, $n, r \geq 0$ and $0 \leq s, t \leq p-1$. Now we may give a q -analogue of (3.1). For $b, c \geq 1$ with $b \mid c$, define

$$T_{c, l}^{(b)}(n, r; q) = \frac{[l]_{q^c}! \Phi_c(q)^l}{[[nb/c]]_{q^{c/b}}!} \sum_{k \equiv r \pmod{c}} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} (k-r)/c \\ l \end{bmatrix}_{q^c},$$

where $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$.

Theorem 3.1. *Let $b \geq 2$ and $n, r, s, t \geq 0$ be integers with $0 \leq s, t \leq b-1$. Suppose that c is a positive multiple of b . Then*

$$T_{bc, l}^{(b)}(bn + s, br + t; q) \equiv (-1)^t q^{\binom{t}{2}} \begin{bmatrix} s \\ t \end{bmatrix}_q T_{c, l}^{(b)}(n, r; q^b) \pmod{\Phi_b(q)}. \quad (3.2)$$

Proof. By the q -Lucas congruence (cf. [4, Proposition 2.2]), we have

$$\begin{bmatrix} bn + s \\ br + t \end{bmatrix}_q \equiv \binom{n}{r} \begin{bmatrix} s \\ t \end{bmatrix}_q \pmod{\Phi_b(q)}. \quad (3.3)$$

Since $q^b \equiv 1 \pmod{\Phi_b(q)}$, (3.3) can be rewritten as

$$\begin{bmatrix} bn + s \\ br + t \end{bmatrix}_q \equiv \begin{bmatrix} n \\ r \end{bmatrix}_{q^b} \begin{bmatrix} s \\ t \end{bmatrix}_q \pmod{\Phi_b(q)}. \quad (3.4)$$

On the other hand, we have

$$(-1)^{bk+t} q^{\binom{bk+t}{2}} \equiv (-1)^{k+t} q^{\binom{t}{2}} \pmod{\Phi_b(q)}. \quad (3.5)$$

In fact, since $q^{\binom{bk+t}{2}} = q^{bk(bk+2t-1)/2 + \binom{t}{2}}$, (3.5) easily follows when b is odd. And if b is even, then

$$q^{b/2} = \frac{1 - q^b}{1 - q^{b/2}} - 1 \equiv -1 \pmod{\Phi_b(q)}.$$

Thus (3.5) is also valid for even b . Since $b \mid c$, it is not difficult to see that $\Phi_{bc}(q) = \Phi_c(q^b)$. Also, $[j]_{q^c} = (1 - q^{jc})/(1 - q^c)$ is prime to $\Phi_b(q)$ for any $j \geq 1$.

Hence,

$$\begin{aligned}
& T_{bc,l}^{(b)}(bn + s, br + t; q) \\
&= \frac{[l]_{q^{bc}}! \Phi_{bc}(q)^l}{[[(bn + s)/c]]_{q^{(bc)/b}}!} \sum_{\substack{bk+t \equiv br+t \\ (\text{mod } bc)}} (-1)^{bk+t} q^{\binom{bk+t}{2}} \begin{bmatrix} bn + s \\ bk + t \end{bmatrix}_q \begin{bmatrix} ((bk + t) - (br + t))/(bc) \\ l \end{bmatrix}_{q^{bc}} \\
&\equiv \frac{[l]_{(q^b)^c}! \Phi_c(q^b)^l}{[[nb/c]]_{(q^b)^c/b}!} \sum_{k \equiv r \pmod{c}} (-1)^{k+t} q^{b\binom{k}{2} + \binom{t}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} \begin{bmatrix} s \\ t \end{bmatrix}_q \begin{bmatrix} (k - r)/c \\ l \end{bmatrix}_{q^{bc}} \\
&= (-1)^t q^{\binom{t}{2}} \begin{bmatrix} s \\ t \end{bmatrix}_q T_{c,l}^{(b)}(n, r; q^b) \pmod{\Phi_b(q)}.
\end{aligned}$$

□

Furthermore, define

$$T_{c,l}^{(b)}(n, r; q, z) = \frac{[l]_{q^c}! \Phi_c(q)^l}{[[nb/c]]_{q^{c/b}}!} \sum_{k \equiv r \pmod{c}} (-1)^k z^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} (k - r)/c \\ l \end{bmatrix}_{q^c}.$$

Then we also have

$$T_{bc,l}^{(b)}(bn + s, br + t; q, z) \equiv (-1)^t z^t q^{\binom{t}{2}} \begin{bmatrix} s \\ t \end{bmatrix}_q T_{c,l}^{(b)}(n, r; q^b, z^b) \pmod{\Phi_b(q)}. \quad (3.6)$$

Below we consider the special case that $s = t = 0$. We need the following q -analogue of the Wolstenholme-Ljunggren congruence.

Lemma 3.1.

$$\begin{bmatrix} bn \\ bm \end{bmatrix}_q \bigg/ \begin{bmatrix} n \\ m \end{bmatrix}_{q^b} \equiv ((-1)^{b-1} q^{\binom{b}{2}})^{(n-m)m} + \frac{(b^2 - 1)nm(n - m)}{24} (1 - q^b)^2 \pmod{\Phi_b(q)^3}. \quad (3.7)$$

Proof. By Andrews' discussions in [1], we have

$$\frac{(q^{jb+1}; q)_{b-1} - (-1)^{j(b-1)} q^{j\binom{b}{2}} (q; q)_{b-1}}{(1 - q^{(j+1)b})(1 - q^{jb})} \equiv \frac{(b^2 - 1)b}{24} \pmod{\Phi_b(q)}, \quad (3.8)$$

though he only proved (3.8) when b is prime. Noting that $(q; q)_{b-1} \equiv b \pmod{\Phi_b(q)}$ and $1 - q^{jb} \equiv j(1 - q^b) \pmod{\Phi_b(q)^2}$, (3.8) can be rewritten as

$$\frac{(q^{jb+1}; q)_{b-1}}{(q; q)_{b-1}} \equiv (-1)^{j(b-1)} q^{j\binom{b}{2}} + \frac{(b^2 - 1)j(j + 1)}{24} (1 - q^b)^2 \pmod{\Phi_b(q)^3},$$

It follows that

$$\begin{aligned} \frac{\begin{bmatrix} bn \\ bm \end{bmatrix}_q}{\begin{bmatrix} n \\ m \end{bmatrix}_{q^b}} &= \frac{\prod_{j=n-m}^{n-1} ((q^{jb+1}; q)_{b-1}/(q; q)_{b-1})}{\prod_{j=0}^{m-1} ((q^{jb+1}; q)_{b-1}/(q; q)_{b-1})} \\ &\equiv (-1)^{(b-1)(n-m)m} q^{\binom{b}{2}(n-m)m} \left(1 + \frac{(b^2-1)nm(n-m)}{24} (1-q^b)^2 \right) \pmod{\Phi_b(q)^3}. \end{aligned}$$

In view of (3.5), we get (3.7). \square

Thus,

$$\begin{aligned} &\frac{[l]_{q^{bc}}! \Phi_{bc}(q)^l}{[[nb/c]]_{q^{bc}}!} \sum_{bk \equiv br \pmod{bc}} (-1)^{bk} z^{bk} q^{\binom{bk}{2}} \begin{bmatrix} bn \\ bk \end{bmatrix}_q \begin{bmatrix} (bk-br)/(bc) \\ l \end{bmatrix}_{q^{bc}} \\ &\equiv \frac{[l]_{(q^b)^c}! \Phi_c(q^b)^l}{[[nb/c]]_{(q^b)^c/b}!} \left((-1)^{(b-1)nr} \sum_{k \equiv r \pmod{c}} (-1)^k z^{bk} q^{nk\binom{b}{2}+b\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} \begin{bmatrix} (k-r)/c \\ l \end{bmatrix}_{q^{bc}} \right. \\ &\quad \left. + \frac{(b^2-1)n}{24} \sum_{k \equiv r \pmod{c}} (-1)^k z^{bk} (1-q^{kb})(1-q^{(n-k)b}) \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} \begin{bmatrix} (k-r)/c \\ l \end{bmatrix}_{q^{bc}} \right) \\ &\pmod{\Phi_b(q)^3}. \end{aligned}$$

That is,

Theorem 3.2. *Suppose that $b, c \geq 2$ and $b \mid c$. Then for $n, r \geq 0$,*

$$\begin{aligned} &\frac{1}{(1-q^b)^2} \cdot (T_{bc,l}^{(b)}(bn, br; q, z) - (-1)^{(b-1)nr} T_{c,l}^{(b)}(n, r; q^b, z^b q^{n\binom{b}{2}})) \\ &\equiv - \frac{[(n-2)b/c]!}{[nb/c]!} \cdot \frac{(b^2-1)n^2(n-1)}{24} \cdot z^b T_{c,l}^{(b)}(n-2, r-1; q^b, z^b) \pmod{\Phi_b(q)}. \end{aligned} \tag{3.9}$$

In particular,

$$\begin{aligned} &T_{b^2,l}^{(b)}(bn, br; q, z) - (-1)^{(b-1)nr} T_{b,l}^{(b)}(n, r; q^b, z^b q^{n\binom{b}{2}}) \\ &\equiv - \frac{(b^2-1)n}{24} \cdot z^b (1-q^b)^2 T_{b,l}^{(b)}(n-2, r-1; q^b, z^b) \pmod{\Phi_b(q)^3}. \end{aligned} \tag{3.10}$$

4. FROM q -CONGRUENCES TO INTEGER CONGRUENCES

In [8], Sun and Davis conjectured that

$$\begin{aligned} &\frac{p^l}{[n/p]!} \sum_{k \equiv r \pmod{p}} (-1)^k \binom{pn}{pk} \left(\frac{k-r}{p} \right)^l \\ &\equiv \frac{p^l}{[n/p]!} \sum_{k \equiv r \pmod{p}} (-1)^k \binom{n}{k} \left(\frac{k-r}{p} \right)^l \pmod{p^3} \end{aligned} \tag{4.1}$$

for prime $p \geq 5$. This conjecture was confirmed by Sun in [7], with help of some arithmetical properties of the Stirling numbers of the second kind.

Define

$$S_{c,l}^{(b)}(n, r; q, z) = \frac{\Phi_c(q)^l}{[[nb/c]]_{q^{c/b}}!} \sum_{k \equiv r \pmod{c}} (-1)^k z^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q [(k-r)/c]_{q^c}^l.$$

Using the similar discussions as above, we also can obtain that

$$S_{bc,l}^{(b)}(bn + s, br + t; q, z) \equiv (-1)^t z^t q^{\binom{t}{2}} \begin{bmatrix} s \\ t \end{bmatrix}_q S_{c,l}^{(b)}(n, r; q^b, z^b) \pmod{\Phi_b(q)}, \quad (4.2)$$

and

$$\begin{aligned} & \frac{1}{(1-q^b)^2} \cdot (S_{bc,l}^{(b)}(bn, br; q, z) - (-1)^{(b-1)nr} S_{c,l}^{(b)}(n, r; q^b, z^b q^{n\binom{b}{2}})) \\ & \equiv - \frac{[(n-2)b/c]!}{[nb/c]!} \cdot \frac{(b^2-1)n^2(n-1)}{24} \cdot z^b S_{c,l}^{(b)}(n-2, r-1; q^b, z^b) \pmod{\Phi_b(q)}. \end{aligned} \quad (4.3)$$

In particular, for prime $p \geq 5$,

$$\begin{aligned} & \frac{[p]_{q^{p^\alpha}}^l}{[[n/p^\alpha]]_{q^{p^\alpha}}!} \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} pn \\ pk \end{bmatrix}_q [(k-r)/p^\alpha]_{q^{p^\alpha}}^l \\ & \equiv \frac{[p]_{q^{p^\alpha}}^l}{[[n/p^\alpha]]_{q^{p^\alpha}}!} \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k q^{nk\binom{p}{2} + p\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^p} [(k-r)/p^\alpha]_{q^{p^\alpha}}^l \\ & + \frac{(p^2-1)n[p]_{q^{p^\alpha}}^l}{24[[n/p^\alpha]]_{q^{p^\alpha}}!} \sum_{k \equiv r \pmod{p^\alpha}} (-1)^k (1-q^{kp})(1-q^{(n-k)p}) \begin{bmatrix} n \\ k \end{bmatrix}_{q^p} [(k-r)/p^\alpha]_{q^{p^\alpha}}^l \\ & \pmod{[p]_q^3}. \end{aligned} \quad (4.4)$$

However, one might doubt whether (4.4) surely implies (4.1), since neither side of (4.4) is a polynomial in q . So we need to give an explanation how to deduce (4.1) from (4.4) by substituting $q = 1$.

Let $L(q)$ and $R(q)$ denote the left side and the right side of (4.4) respectively. Let

$$F(q) = \frac{[[n/p^\alpha]]_{q^{p^\alpha}}!}{\prod_{j \geq \alpha+1} [p]_{q^{p^{j-1}}}^{[n/p^j]}.$$

Then clearly $F(q) \in \mathbb{Z}[q]$ in view of (2.2). And from Corollary 2.1, we also know that $F(q)L(q), F(q)R(q) \in \mathbb{Z}[q]$. Hence there exists a polynomial $H(q) \in \mathbb{Z}[q]$ such that

$$F(q)L(q) - F(q)R(q) = [p]_q^3 H(q).$$

Substituting $q = 1$, we get

$$F(1)L(1) \equiv F(1)R(1) \pmod{p^3}.$$

But by (1.7),

$$F(1) = \frac{[n/p^\alpha]!}{p^{\sum_{j \geq \alpha+1} [n/p^j]}}$$

is not divisible by p . Thus (4.1) is concluded.

Acknowledgment. I am grateful to Professor Zhi-Wei Sun for his helpful suggestions on this paper.

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